Symbolic Defect of Monomial Ideals

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Background

Definition. Let R be commutative Noetherian, and let I be an ideal of R. The *n-th symbolic power of I*, denoted $I^{(n)}$, is

$$I^{(n)} = \bigcap_{\mathfrak{p}\in\operatorname{Ass}(I)} I^n R_{\mathfrak{p}} \cap R.$$

Two important properties:

1. $I^n \subseteq I^{(n)}$ for all n
2. Symbolic powers form a graded family of ideals

The *Rees algebra* $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n t^n$ is Noetherian, but the *Symbolic Rees algebra* $\mathcal{R}_s(I) = \bigoplus_{n=0}^{\infty} I^{(n)} t^n$ may not be.

Assume R is local (resp. graded) with unique maximal (resp. irrelevant) ideal $\mathfrak{m}$. The *analytic spread* and the *symbolic analytic spread* of I are

$$\ell(I) = \dim\left(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)\right) \quad \text{and} \quad s\ell(I) = \dim\left(\mathcal{R}_s(I)/\mathfrak{m}\mathcal{R}_s(I)\right).$$

Letting $\mu(M)$ denote the minimal number of generators of an R-module M. We also obtain that $\ell(I)$ and $s\ell(I)$ are one more than the growth rate of the functions, $n \mapsto \mu(I^n)$ and $n \mapsto \mu(I^{(n)})$, respectively.

Definition (Galetto-Geramita-Shin-Van Tuyl, 2019)**.** The *symbolic defect function* of an ideal I is the numerical function

$$\operatorname{sdef}_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}, \quad \operatorname{sdef}_I(n) := \mu\left(I^{(n)}/I^n\right) = \dim_K\left(\frac{I^{(n)}}{I^n + \mathfrak{m}I^{(n)}}\right).$$

The symbolic defect acts as a measurement of "closeness" between I^n and $I^{(n)}$. A result from Drabkin states the following:

Theorem (Drabkin 2020)**.** Let $I \subseteq R$ be a homogeneous ideal of a Noetherian graded ring with Noetherian symbolic Rees algebra. Then $\operatorname{sdef}_I(n)$ is eventually quasi-polynomial for $n \gg 0$, with quasi-period $\operatorname{lcm}(d_1, \ldots, d_s)$, where $d_1, \ldots, d_s$ are the degrees of the generators of $\mathcal{R}_s(I)$ as an R-algebra.

Symbolic Powers and Monomial Ideals

Let $R = K[x_1, \ldots, x_r]$, I a monomial ideal. Some nice results are known for monomial ideals:

1. If $r = 2$, then $I^n = I^{(n)}$ for all n.
2. If I is associated to the maximal ideal, then $I^n = I^{(n)}$ for all n.
3. The symbolic Rees algebra is Noetherian.

With Drabkin's theorem, we can conclude that $\operatorname{sdef}_I(n)$ is eventually quasi-polynomial.

We also have a formulation to calculate symbolic powers:

Lemma 1 (Herzog-Hibi-Viêt Trung, 2007)**.** Let I be a monomial ideal in $K[x_1, \ldots, x_r]$ with monomial primary decomposition, $I = Q_1 \cap \cdots \cap Q_s$. Set $\max(I)$ to be the set of maximal associated primes, and, for each $\mathfrak{p} \in \max(I)$, let $Q_{\subseteq\mathfrak{p}} = \bigcap_{\sqrt{Q_i}\subseteq\mathfrak{p}} Q_i$. Then,

$$I^{(n)} = \bigcap_{\mathfrak{p}\in\max(I)} (Q_{\subseteq\mathfrak{p}})^n.$$

In particular, if I does not have embedded primes, then $I^{(n)} = Q_1^n \cap \cdots \cap Q_s^n$.

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Convex Polyhedra and Monomial Ideals

We can study monomial ideals and powers thereof through the lens of convex geometry:

Definition. Let $R = k[x_1, \ldots, x_r] = k[\mathbf{x}]$, and let I be a monomial ideal. We define the *Newton polyhedron of I*, denoted $\operatorname{NP}(I)$, as the convex hull of exponent vectors for monomials in I:

$$\operatorname{NP}(I) = \operatorname{cvxhull}\{\mathbf{b} \in \mathbb{Z}^n_{\geq 0} : \mathbf{x}^{\mathbf{b}} \in I\}.$$

A point $(u_1, \ldots, u_r)$ is in $\operatorname{NP}(I)$ if and only if $x_1^{u_1} \cdots x_r^{u_r} \in \overline{I}$. Also, $\operatorname{NP}(I^n) = n\operatorname{NP}(I)$.

We define a similar polyhedron for symbolic powers.

Definition. [Camarneiro et al., 2022] Let I and Q_i be as in Lemma 1. Then, the *symbolic polyhedron* of I, denoted $\operatorname{SP}(I)$, is

$$\operatorname{SP}(I) = \bigcap_{\mathfrak{p}\in\max(I)} \operatorname{NP}(Q_{\subseteq\mathfrak{p}}).$$

In particular, if I has no embedded primes, then $\operatorname{SP}(I) = \operatorname{NP}(Q_1) \cap \cdots \cap \operatorname{NP}(Q_s)$.

Similarly, a point $(u_1, \ldots, u_r)$ is in $\operatorname{SP}(I)$ if and only if $x_1^{u_1} \cdots x_r^{u_r} \in \overline{I^{(n)}}$ for some n. Also, $\operatorname{SP}(I^{(n)}) = n\operatorname{SP}(I)$.

There is also an interpretation of analytic spread and symbolic analytic spread in this context.

Theorem (Há, Nguyễn 2021)**.** For a polyhedron Δ, let $\operatorname{mdc}(\Delta)$ denote the maximal dimension of a compact facet of Δ. Then $\ell(I) = \operatorname{mdc}(\Delta) + 1$, $s\ell(I) = \operatorname{mdc}(\Delta) + 1$.

Example: $I = (xy, xz, yz)$

Note that $I = (x, y) \cap (x, z) \cap (y, z)$. For each n

$$I^{(n)} = (x, y)^n \cap (x, z)^n \cap (y, z)^n.$$

Note that $I^2 = (x^2y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2)$, and $I^{(2)} = (x^2y^2, x^2z^2, y^2z^2, xyz)$. Since xyz is the only minimal generator of $I^{(2)}$ not in I^2, $\operatorname{sdef}_I(2) = 1$.

Inductively, we can find an exact formula:

$$\operatorname{sdef}_I(n) = \begin{cases} \frac{3}{2}n - 2 & n \equiv 0 \pmod 2 \\ \frac{3}{2}n - \frac{3}{2} & n \equiv 1 \pmod 2 \end{cases}.$$

Below, we see the respective polyhedra for I:

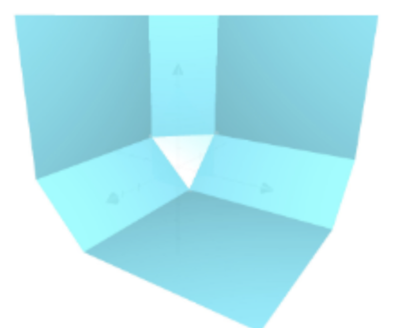


FIGURE 1. $\operatorname{NP}(xy, yz, xz)$

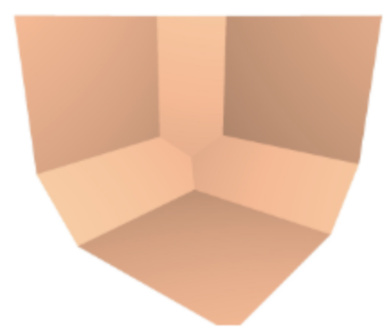


FIGURE 2. $\operatorname{SP}(xy, yz, xz)$

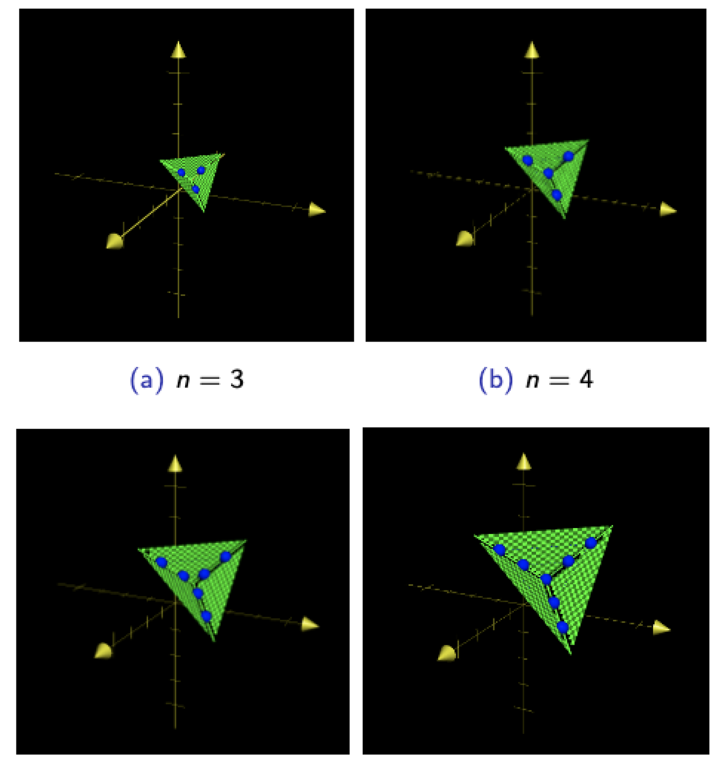
$$\operatorname{NP}(I) = \begin{cases} u, v, w \geq 0 \\ u + v \geq 2 \\ v + w \geq 2 \\ u + w \geq 2 \\ u + v + w \geq 3 \end{cases} \qquad \operatorname{SP}(I) = \begin{cases} u, v, w \geq 0 \\ u + v \geq 2 \\ v + w \geq 2 \\ u + w \geq 2 \end{cases}$$

Notice that $\operatorname{mdc}(\operatorname{NP}(I)) = 2$, corresponding to the triangular face supported on $u + v + w = 3$. Thus, $\ell(I) = 3$. However, $\operatorname{mdc}(\operatorname{SP}(I)) = 1$, so $s\ell(I) = 2$.

Symbolic Defect Calculation

For an ideal with integrally closed powers and symbolic powers, we can count the "minimal" points of $n\operatorname{SP}(I)$ not in $n\operatorname{NP}(I)$ to calculate symbolic defect.

Below, we see an example of this for smaller cases, using the example of $I = (xy, xz, yz)$:



(a) $n = 3$ (b) $n = 4$

(c) $n = 5$ (d) $n = 6$

Example (**O**. 2023)**.** Let $I = (x^a y, y^b z, z^c x, xyz)$. Then,

1. $\operatorname{sdef}_I(n) \sim (\alpha + \beta + \gamma)n$, where $P = (\alpha, \beta, \gamma)$ solves $\begin{cases} \frac{1}{a}u + v = 1 \\ \frac{1}{b}v + w = 1 \\ \frac{1}{c}w + u = 1 \end{cases}$.
2. A quasi-period of $\operatorname{sdef}_I(n)$ is $abc + 1$.

Integral Symbolic Defect

Counting minimal points of $n\operatorname{SP}(I)$ not in $n\operatorname{NP}(I)$ does not calculate $\operatorname{sdef}_I(n)$ in general, since powers and/or symbolic powers of I may not be integrally closed. Consider instead:

Definition. Let I be an ideal of a commutative ring R. We define the *integral symbolic defect of I* as

$$\operatorname{isdef}_I(n) := \mu\left(\overline{I^{(n)}}/\overline{I^n}\right) = \dim_K\left(\frac{\overline{I^{(n)}}}{\overline{I^n} + \mathfrak{m}\overline{I^{(n)}}}\right).$$

Theorem (**O**. 2023)**.** Let I be a monomial ideal in $R = K[x_1, \ldots, x_r]$ such that $I^{(n)} = (I^n)^{\text{sat}}$ for all $n \geq 1$ (e.g., when $\dim(R/I) = 1$). Then $\operatorname{isdef}_I(n) = O(n^{r-2})$.

The proof involves reducing to an $(r-2)$-dimensional polytope, and invoking the theory of Ehrhart polynomials.

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